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UNSTABLE TRAVELLING WAVES IN THE FRICTION-INDUCED VIBRATION OF DISCS

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1. INTRODUCTION

There are many mechanical devices that involve a disc with an auxiliary system; examples include car disc brakes, computer discs and circular saws. Pioneering research into the dynamic instability of such systems started with the study of a stationary disc excited by a rotating load [1–3] and a disc spinning past a stationary load [4]. In all cases the system could become unstable at some specific values of system parameters (typically mass, stiffness and damping) and running conditions even when friction between the two contacting components in relative rotation was absent. Hutton *et al.* [5] studied the frequency speed characteristics of guided rotating discs affected by the spring stiffness, the number of springs and spring location using a Galerkin method. Yu and Mote [6] investigated the parametric vibration of circular plates containing small imperfections through a perturbation method. Jiang *et al.* [7] analyzed the dynamic response of a floppy disc system under axial excitation. Shen and Mote [8, 9] gave a mechanism for the instability of a circular plate caused by the damper in the rotating mass–spring–damper system in the supercritical speed range and looked at the stability of asymmetric circular plates. Shen [10] studied the parametric resonances of a stationary disc excited by a rotating slider using the method of multiple scales. Huang and Mote [11] considered a large damping force acting on a spinning disc.

The modelling of friction as a follower force has a considerable history in the literature on disc vibrations. Ono *et al.* [12] included follower-force friction in their study of instabilities in computer disc-drives and found that waves travelling in the same direction as the disc were rendered unstable. Tian and Hutton [13] used a follower force (or a regenerative force in their terminology) in the modelling of wood-saw dynamics. Lee and Waas [14] examined the instability of a rotating annular laminated brake disc under a non-rotating frictional follower force. Hulten [15] considered different treatments of friction (including the follower force model) for drum-brake analysis. Chan *et al.* [16] found that the rotating frictional follower force made the backward travelling waves unstable in the special case of the resonances remote from combination resonances of a stationary disc. Tseng and Wickert [17] considered friction to be distributed over a sector and treated it as a follower force. Mottersheard *et al.* [18] treated the brake pad as an elastic foundation (with mass, damping and follower-force friction). Ouyang *et al.* [19] investigated the dynamic instability of a stationary circular plate excited by a rotating friction force with negative slope, and in a further paper [20] developed a general method for analyzing the dynamic behaviour of a car disc brake system.

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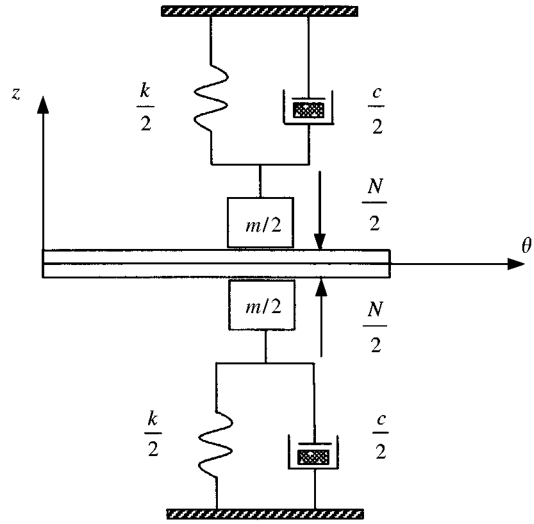

Figure 1. Simple representation of the floating calliper disc brake system.

The floating-calliper disc brake system can be represented very simply by the arrangement shown in Figure 1. The dynamic load $m\ddot{x} + c\dot{x} + kx$ is superimposed on the initial stress problem of the normal load $N/2$ due to the applied brake pressure. Straightforward analysis shows that the disc is subjected to a transverse dynamic load, together with a constant frictional load of μN on the disc contact surfaces opposite to the direction of rotation of the disc. Transverse vibration modes of the disc (and vibration modes of the other brake components) occur in the range of the squeal frequencies (typically 2–4 kHz), in which case the brake pads follow the deflections of the disc and the friction forces have small components in the transverse direction.

The study and prevention of unstable vibration is very important to the vehicle brakes industry and there is an urgent need for a model that will predict unstable squeal-noise dynamics with reasonable accuracy. The follower-force model is a candidate and its suitability will depend upon the quality of its predictions when compared to experimental observations. The follower-force model can be justified for practical application if it is capable of reproducing observed behaviour in physical systems. Dynamic measurements from systems containing dry friction are notoriously difficult to obtain, not only because the data from displacement (or acceleration) transducers tend to be very noisy but also because the system can change significantly with wear on the surfaces. One aspect of this latter point is that squeal frequency experiments are not very repeatable. However, the existence of backward and forward travelling waves is thought to be relatively straightforward to confirm [21], and is the motivation for the present theoretical study. The previous paper of Chan *et al.* [16] is extended to include the direction of the travelling waves at combination-type instabilities for the first time.

2. VIBRATION OF A STATIONARY DISC UNDER A ROTATING FRICTION LOAD

The equation of transverse motion of a stationary disc excited by a rotating mass–spring–damper system with dry friction, in a cylindrical co-ordinate system shown in

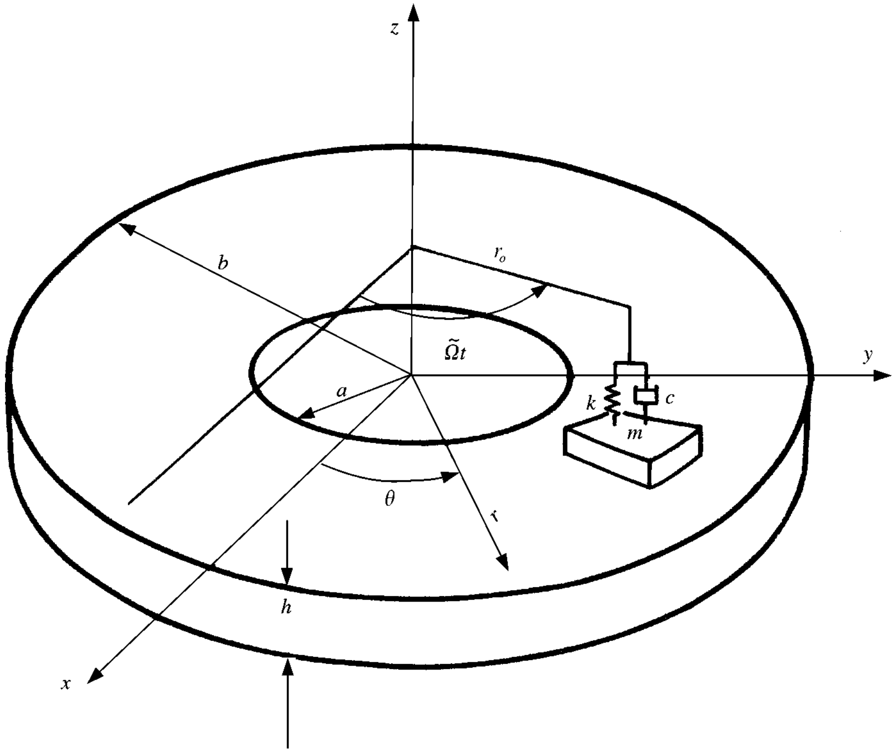

Figure 2. The co-ordinate system of the disc.

Figure 2, may be written as

$$\rho h \frac{\partial^2 w}{\partial t^2} + D^* \nabla^4 \dot{w} + D \nabla^4 w = -\frac{1}{r}\,\delta(r - r_0)\,\delta(\theta - \tilde{\Omega}t)\left[m\left(\frac{\partial}{\partial t} + \tilde{\Omega}\frac{\partial}{\partial \theta}\right)^2 + c\left(\frac{\partial}{\partial t} + \tilde{\Omega}\frac{\partial}{\partial \theta}\right) + k - F\frac{\partial}{r\partial \theta}\right] w, \quad (1)$$

for which a list of notation is given in Appendix A.

The transverse motion of the disc can be expressed in modal co-ordinates such that,

$$w(r, \theta, t) = \sum_{k=0}^{\infty} \sum_{l=-\infty}^{\infty} \psi_{kl}(r, \theta)\, q_{kl}(t), \quad (2)$$

where,

$$\psi_{kl}(r, \theta) = \frac{1}{\sqrt{\rho h b^2}} R_{kl}(r) \exp(\mathrm{i} l \theta), \quad (3)$$

The modal functions satisfy the ortho-normality conditions,

$$\int_a^b \rho h\, \bar{\psi}_{kl} \psi_{rs} r \mathrm{d}r \mathrm{d}\theta = \delta_{kr}\, \delta_{ls},$$

$$\int_a^b D\, \bar{\psi}_{kl} \nabla^4 \psi_{rs} r \mathrm{d}r \mathrm{d}\theta = \omega_{rs}^2\, \delta_{kr}\, \delta_{ls}, \quad (4)$$

where the overbar denotes complex conjugation.

Substituting equations (2) and (3) into equation (1), multiplying by $\bar{\psi}_{rs}$, integrating over the disc area and then using equation (4) leads to [16],

$$\ddot{q}_{kl} + 2\xi\omega_{kl}^2\dot{q}_{kl} + \omega_{kl}^2 q_{kl} = -\frac{1}{\rho h b^2}\sum_{r=0}^{\infty}\sum_{s=-\infty}^{\infty} R_{rs}(r_0)\,R_{kl}(r_0)\exp\left[\mathrm{i}(s-l)\tilde{\Omega}\,t\right]$$
$$\times\left[m(\ddot{q}_{rs} + \mathrm{i}2s\tilde{\Omega}\dot{q}_{rs} - s^2\,\tilde{\Omega}^2\,q_{rs}) + c(\dot{q}_{rs} + \mathrm{i}s\tilde{\Omega}q_{rs}) + \left(k - \frac{\mathrm{i}sF}{r_0}\right)q_{rs}\right]. \tag{5}$$

Equation (5) is an infinite system of Hill's equations. Therefore, only an approximate solution may be found. The dynamic behaviour of equation (5) is determined by the system parameters and operating conditions and may become unstable. The disc can be unstable at some particular values of the system parameters and operating conditions. In the subcritical speed range, instability is caused by the friction [16]. In the supercritical speed range, instability is more pronounced because damping may become destabilizing even when friction is absent [10]. The instability thus caused is also referred to as a parametric resonance since it is caused by cyclical variation of the system parameters and not by an external applied load.

3. MULTIPLE SCALES ANALYSIS

When any one of the parameters of the rotating mass–spring–damper systems is very small, a perturbation method may be used, which can reduce the amount of computation that would be necessary by using other methods such as the state space method. The method of multiple scales [22] is particularly suitable for solving the above problem, and similar problems [10, 16]. The following new variables are introduced,

$$\tau = \omega_{cr}t, \quad \beta_{kl} = \frac{\omega_{kl}}{\omega_{cr}}, \quad \Omega = \frac{\tilde{\Omega}}{\omega_{cr}}, \tag{6}$$

where

$$\omega_{cr} = \min\left[\frac{\omega_{kl}}{l}; \quad k = 0, 1, 2, \ldots; \quad l = 1, 2, \ldots\right] \tag{7}$$

and the parameter ε is used to signify a small quantity. It is assumed that the mass, damping, stiffness and friction are all small,

$$\varepsilon\gamma = \frac{m}{\rho h b^2}, \quad \varepsilon\varsigma = \frac{c}{\rho h b^2\omega_{cr}}, \quad \varepsilon\kappa = \frac{k}{\rho h b^2\omega_{cr}^2}, \quad \varepsilon f = \frac{F}{\rho h b^2\,\omega_{cr}^2 r_0}, \quad \varepsilon\xi = \xi_K\omega_{cr}. \tag{8}$$

Substitution of equations (6)–(8) into equation (5) yields

$$\frac{\mathrm{d}^2 q_{kl}}{\mathrm{d}\tau^2} + 2\varepsilon\xi\beta_{kl}^2\frac{\mathrm{d}q_{kl}}{\mathrm{d}\tau} + \beta_{kl}^2 q_{kl} = -\sum_{r=0}^{\infty}\sum_{s=-\infty}^{\infty} R_{rs}(r_0)R_{kl}(r_0)\exp\left[\mathrm{i}(s-l)\Omega\tau\right]$$
$$\times\left\{\varepsilon\gamma\left(\frac{\mathrm{d}}{\mathrm{d}\tau} + \mathrm{i}s\Omega\right)^2 q_{rs} + \varepsilon\varsigma\left(\frac{\mathrm{d}}{\mathrm{d}\tau} + \mathrm{i}s\Omega\right)q_{rs} + \varepsilon(\kappa - \mathrm{i}sf)\,q_{rs}\right\}. \tag{9}$$

Independent time scales are defined as multiples of the integer powers of ε,

$$T_0 = \tau, \quad T_1 = \varepsilon\tau, \quad T_2 = \varepsilon^2\tau, \dots, \tag{10}$$

and a solution q_{kl} is sought in the form,

$$q_{kl} = q_{kl}^{(0)} + \varepsilon q_{kl}^{(1)} + \varepsilon^2 q_{kl}^{(2)} + \cdots \tag{11}$$

The usual procedure is to separate equation (9) into a number of equations defined by selecting the coefficients of the different powers of ε. The solution of the zeroth order equation is

$$q_{kl}^{(0)} = A_{kl}(T_1)\exp(\mathrm{i}\beta_{kl}T_0) + B_{kl}(T_1)\exp(-\mathrm{i}\beta_{kl}T_0), \tag{12}$$

where A_{kl} and B_{kl} are determined from the equation of first order terms. This latter equation can be arranged in the form,

$$\begin{aligned}(D_0^2 + 2\xi\beta_{kl}^2 D_0 + \beta_{kl}^2)q_{kl}^{(1)} = &-\mathrm{i}2\beta_{kl}(D_1 + \xi\beta_{kl}^2)[A_{kl}\exp(\mathrm{i}\beta_{kl}\tau) - B_{kl}\exp(-\mathrm{i}\beta_{kl}\tau)]\\ &-\sum_{r=0}^{\infty}\sum_{s=-\infty}^{\infty} R_{rs}(r_0)R_{kl}(r_0)\exp[\mathrm{i}(s-l)\Omega\tau]\\ &\times[D_{rs}^{+}A_{rs}\exp(\mathrm{i}\beta_{rs}\tau) + D_{rs}^{-}B_{rs}\exp(-\mathrm{i}\beta_{rs}\tau)],\end{aligned} \tag{13}$$

where

$$D_0 = \frac{\mathrm{d}}{\mathrm{d}T_0}, \quad D_1 = \frac{\mathrm{d}}{\mathrm{d}T_1},$$

$$\begin{aligned}D_{rs}^{+} &= \kappa - \gamma(C_{rs}^{+})^2 + \mathrm{i}[\varsigma C_{rs}^{+} - sf] \quad C_{rs}^{+} = \beta_{rs} + s\Omega,\\ D_{rs}^{-} &= \kappa - \gamma(C_{rs}^{-})^2 - \mathrm{i}[\varsigma C_{rs}^{-} + sf] \quad C_{rs}^{-} = \beta_{rs} - s\Omega.\end{aligned} \tag{14}$$

It should be noticed that $C_{rs}^{-} > 0$ in the subcritical speed range but $C_{rs}^{-} < 0$ may be true in the supercritical range.

Of course the assumption of smallness in the parameters of equation (8) is a serious one which has consequences for the accurate location of the regions of unstable disc dynamics which is the main purpose of the analysis. However, Chan *et al.*, [16] using typical parameters of a disc brake found the instability regions determined from multiple scales analysis to be similar to those obtained by the state-space method, which does not assume small parameters but requires significantly more computation.

Previous work [10, 16] indicates that combination resonances occur close to,

$$(s \pm l)\Omega = \beta_{rs} \pm \beta_{kl}\,(s > l,\ l \geqslant 0) \tag{15}$$

and that single-mode resonances will be close to

$$2l\Omega = 2\beta_{kl}\,(l > 0). \tag{16}$$

The transverse vibration of the disc can be expressed in the form of a series of standing waves and forward and backward travelling waves as,

$$\begin{aligned}w(r,\theta,t) &= \frac{1}{\sqrt{\rho h b^2}}\sum_{k=0}^{\infty}\sum_{l=-\infty}^{\infty} R_{kl}(r)\exp(\mathrm{i}l\theta)q_{kl}(t) = \frac{1}{\sqrt{\rho h b^2}}\sum_{k=0}^{\infty}[R_{k0}(r)q_{k0}(t)\\ &+\sum_{l=1}^{\infty} R_{kl}(r)(\{A_{kl}\exp[\mathrm{i}(l\theta + \beta_{kl}\tau)] + \mathrm{cc}\} + \{B_{kl}\exp[\mathrm{i}(l\theta - \beta_{kl}\tau)] + \mathrm{cc}\})] + O(\varepsilon),\end{aligned} \tag{17}$$

where cc denotes the complex conjugate.

From equation (17), it is clear that if and only if $A_{kl}(T_1)$ (or $B_{kl}(T_1)$) is divergent then the backward (or forward) travelling wave of the (*k*, *l*) mode is made unstable. Therefore, the subsequent analysis on the stability of travelling waves is conducted by examining $A_{kl}(T_1)$ and $B_{kl}(T_1)$ in all possible resonances in the whole speed range for all system parameters of interest. Note that the standing wave modes represented by $R_{kl}(r)$ will not be destabilized [16] and, therefore, are not considered in the subsequent analysis.

4. TRAVELLING WAVES IN COMBINATION RESONANCES

There are four types of combination resonances, as shown in equation (15) by the different combinations of plus and minus signs. For simplicity, the mathematical treatment is presented only for the following type,

$$(s + l)\Omega = \beta_{rs} - \beta_{kl} \; (s > l, \; l \geqslant 0),$$

which occurs normally at low speeds. In the range close to this resonance one may write,

$$(s + l)\Omega = \beta_{rs} - \beta_{kl} + \varepsilon\sigma \; (s > l, \; l \geqslant 0), \tag{18}$$

where σ is a detuning parameter.

To ensure that the terms in expansion (11) really do have increasing orders of smallness, it is necessary to set certain secular terms in equation (13) to zero. This leads to the following expressions, for the (*k*, *l*) mode,

$$-\mathrm{i}2\beta_{kl}(D_1 + \xi\beta_{kl}^2)\,A_{kl} - R_{kl}^2(r_0)\,D_{kl}^{+}A_{kl} - R_{rs}(r_0)\,R_{kl}(r_0)\,D_{r,-s}^{+}\,A_{r,-s}\exp(-\mathrm{i}\sigma T_1) = 0,$$

$$\mathrm{i}2\beta_{kl}(D_1 + \xi\beta_{kl}^2)\,B_{kl} - R_{kl}^2(r_0)\,D_{kl}^{-}\,B_{kl} = 0 \tag{19}$$

and for the (*r*, *s*) mode,

$$-\mathrm{i}2\beta_{rs}(D_1 + \xi\beta_{rs}^2)\,A_{rs} - R_{rs}^2(r_0)\,D_{rs}^{+}A_{rs} = 0,$$

$$\mathrm{i}2\beta_{rs}(D_1 + \xi\beta_{rs}^2)\,B_{rs} - R_{rs}^2(r_0)\,D_{rs}^{-}B_{rs} - R_{kl}(r_0)\,R_{rs}(r_0)\,D_{k,-l}^{-}\,B_{k,-l}\exp(-\mathrm{i}\sigma T_1) = 0. \tag{20}$$

The second equation in equations (19) and the first equation in equations (20) are ordinary differential equations and can be solved straightforwardly. The first equation of equations (19) and the second equation of equations (20) form a pair of simultaneous ordinary differential equations. Since $q_{k,-l} = \bar{q}_{kl}$ ($q_{r,-s} = \bar{q}_{rs}$), it follows that $A_{r,-s} = \bar{B}_{rs}$ and $B_{k,-l} = \bar{A}_{kl}$. The solution of equations (19) and (20) is

$$A_{kl} = G_{kl}\exp(\mathrm{i}\lambda T_1),$$

$$B_{kl} = H_{kl}\exp\left\{\left[-\frac{lf + \varsigma C_{kl}^{-}}{2\,\beta_{kl}}\,R_{kl}^2(r_0) - \xi\beta_{kl}^2 - \frac{\mathrm{i}R_{kl}^2(r_0)\,[\kappa - \gamma(C_{kl}^{-})^2]}{2\beta_{kl}}\right]T_1\right\},$$

$$A_{rs} = G_{rs}\exp\left\{\left[\frac{sf - \varsigma C_{rs}^{+}}{2\,\beta_{rs}}\,R_{rs}^2(r_0) - \xi\beta_{rs}^2 + \frac{\mathrm{i}R_{rs}^2(r_0)\,[\kappa - \gamma(C_{rs}^{+})^2]}{2\beta_{rs}}\right]T_1\right\},$$

$$B_{rs} = H_{rs}\exp[-\mathrm{i}(\bar{\lambda} + \sigma)\,T_1], \tag{21}$$

where the characteristic exponent λ is determined by solving the following quadratic equation of complex coefficients,

$$[2\beta_{rs}(\lambda+\sigma) - \mathrm{i}2\xi\beta_{rs}^3 - R_{rs}^2(r_0)\bar{D}_{rs}^-][2\beta_{kl}\lambda - \mathrm{i}2\xi\beta_{kl}^3 - R_{kl}^2(r_0)D_{kl}^+] - R_{rs}^2(r_0)\bar{D}_{rs}^- R_{kl}^2(r_0)D_{kl}^+ = 0. \tag{22}$$

The real part of the exponents in equation (21) determines if a wave is stable or not. If a particular parameter is seen to make a positive contribution to the real part of an exponent, then this parameter is destablizing towards the wave represented by that exponent. If its contribution is negative, then it is stabilizing. In this light, the solution in equation (21) indicates that damping of the disc is always stabilizing. The damping of the rotating system is seen to be stabilizing in the subcritical range but is destabilizing some (k, l) modes and the associated forward travelling waves (when $C_{kl}^- < 0$ for some l in the supercritical speed range so that ς makes a positive contribution to the exponent of B_{kl}). Shen and Mote [8] explained the reason why damping of the rotating system could be destablizing in the supercritical speed range.

The friction is obviously destabilizing the backward (r, s) waves (see A_{rs} in equation (21)). It is also known (by solving equation (22) for λ) that the friction makes A_{kl} and B_{rs} divergent at certain conditions [16] when the imaginary part of a λ is negative. In summary, the friction can destabilize both the forward and backward travelling waves of the (r, s) mode but only the backward travelling wave of the (k, l) mode for the combination resonances of $(s+1)\Omega = \beta_{rs} - \beta_{kl}$ $(s > l, l \geqslant 0)$. The regions of instability for these types of resonances are determined by certain values of system parameters and the operating parameter.

Examination of equation (21) reveals the overall direction of the unstable travelling waves in terms of the rotating speed of the load. Since the real parts of $\mathrm{i}\lambda$ in the exponent of A_{kl} and $-\mathrm{i}\bar{\lambda}$ in the exponent of B_{rs} are equal, no conclusion can be made about the overall direction of travelling wave from these two coefficients. However, since the friction makes a positive contribution to the exponent of A_{rs} but a negative one to that of B_{kl}, it can be concluded that the overall trend, when unstable dynamic behaviour occurs, is for the travelling waves to be dominated by backward motion. As equation (18) indicates that for this kind of combination resonances the values of Ω are small, one may conclude that at low speed range, both backward and forward travelling waves may be destabilized by friction but the dominant wave motion will be backward.

For the resonances of $(s - l)\,\Omega = \beta_{rs} - \beta_{kl}$ $(s > l, l \geqslant 0)$, it can be derived that,

$$\begin{aligned}
A_{kl} &= G_{kl}\exp\left\{\left[\frac{lf - \varsigma C_{kl}^-}{2\,\beta_{kl}} R_{kl}^2(r_0) - \xi\beta_{kl}^2 + \frac{\mathrm{i}R_{kl}^2(r_0)[\kappa - \gamma(C_{kl}^+)^2]}{2\beta_{kl}}\right] T_1\right\},\\
B_{kl} &= H_{kl}\exp(\mathrm{i}\lambda T_1),\\
A_{rs} &= G_{rs}\exp\left\{\left[\frac{sf - \varsigma C_{rs}^+}{2\,\beta_{rs}} R_{rs}^2(r_0) - \xi\beta_{rs}^2 + \frac{\mathrm{i}R_{rs}^2(r_0)[\kappa - \gamma(C_{rs}^+)^2]}{2\beta_{rs}}\right] T_1\right\},\\
B_{rs} &= H_{rs}\exp[\mathrm{i}(\lambda - \sigma)T_1],
\end{aligned} \tag{23}$$

where the characteristic exponent λ is determined by the following equation

$$[2\beta_{rs}(\lambda-\sigma) - \mathrm{i}2\xi\beta_{rs}^3 + R_{rs}^2(r_0)D_{rs}^-][2\beta_{kl}\lambda - \mathrm{i}2\xi\beta_{kl}^3 + R_{kl}^2(r_0)D_{kl}^-] - R_{rs}^2(r_0)D_{rs}^- R_{kl}^2(r_0)D_{kl}^- = 0. \tag{24}$$

It can be seen that damping of the disc is always stabilizing, that damping of the rotating system is stabilizing in the subcritical speed range and can destabilize only the forward travelling waves at certain conditions in the supercritical speed range, and that the friction makes A_{kl} and A_{rs} divergent. Thus, it is clear that the friction can destabilize both the forward and backward travelling waves of the (k, l) and (r, s) modes for the resonances type of $(s - l)\Omega = \beta_{rs} - \beta_{kl}$ $(s > l, l \geqslant 0)$.

But by examining equation (23) in a similar way to examining equation (21), one cannot ascertain the dominant direction of the travelling waves without performing specific calculations. Both travelling waves can be made unstable at this relatively high-speed range, which is well outside the range of interest for disc brake design.

Similarly, the influence of damping, stiffness and the friction on the instability of the travelling waves for the other two types of combination resonances can be found but are not presented here. The instability of these two types of combination resonances occur in the supercritical speed range.

Single mode resonances of the form $2l\Omega + 2\beta_{kl} = \varepsilon\sigma$ $(l > 0)$ occur at or above the first critical speed of the disc. Multiple scales analysis (not included in this paper) can be used to show that for all the single-mode resonances backward waves are destabilized, and forward waves stabilized, by friction. The forward waves are destabilized by the stiffness of the rotating system, for all the single-mode resonances. For certain speed-independent resonances the backward waves are destabilized by friction whilst the forward waves are stabilized. Chan *et al.* [16] studied the speed independent resonances but did not include damping in the disc or in the rotating system.

5. A SIMULATED EXAMPLE

The above analysis on different types of resonances shows that the travelling waves which are destabilized by the friction depend on the types of resonances concerned and on the specific values of system and operating parameters. To offer a whole picture about the instability of the travelling waves, an example is presented here.

The dimensions and properties of the disc are, $a = 0{\cdot}067$ m, $b = 0{\cdot}12$ m, $h = 0{\cdot}01$ m, $E = 120$ GPa, $v = 0{\cdot}25$, $\rho = 7000\ \mathrm{kg\,m^{-3}}$, $\varepsilon\xi = 0$, and for the rotating system, $r_0 = 0{\cdot}1$ m, $\varepsilon\gamma = 0{\cdot}24$, $\varepsilon\varsigma = 5 \times 10^{-5}$, $\varepsilon\kappa = 0{\cdot}7$. εf can vary between 0 and 5×10^{-5}. The disc is clamped at the inner radius and free at the outer radius. The first five frequencies of the disc are $\omega_{kl} = 14270$, 14630, 16070, 19370 and 25100 (rad/s) for $k = 0$, $l = 0, 1, 2, 3, 4$ respectively.

The regions of instability induced by the friction for the summation- and difference-type combination resonances ($(s + l)\Omega = \beta_{rs} - \beta_{kl}$ and $(s - l)\Omega = \beta_{rs} - \beta_{kl}$ $(s > l, l \geqslant 0)$) are illustrated in Figure 3. They all occur at subcritical speeds. The instability of the forward and backward travelling waves is summarized in Table 1. The regions of instability which are located in the supercritical range are not presented since they are unimportant to a large class of engineering systems including disc brakes.

TABLE 1

Instability of travelling waves for combination resonances in the subcritical speed range

Combination resonance types	Modes	Forward wave	Backward wave
$(s + l)\Omega = (\beta_{rs} - \beta_{kl})$	(r, s)	Unstable	Stable*
	(k, l)	Stable	Unstable ⇐
$(s - l)\Omega = (\beta_{rs} - \beta_{kl})$	(r, s)	Unstable	Stable*
	(k, l)	Unstable	Stable*

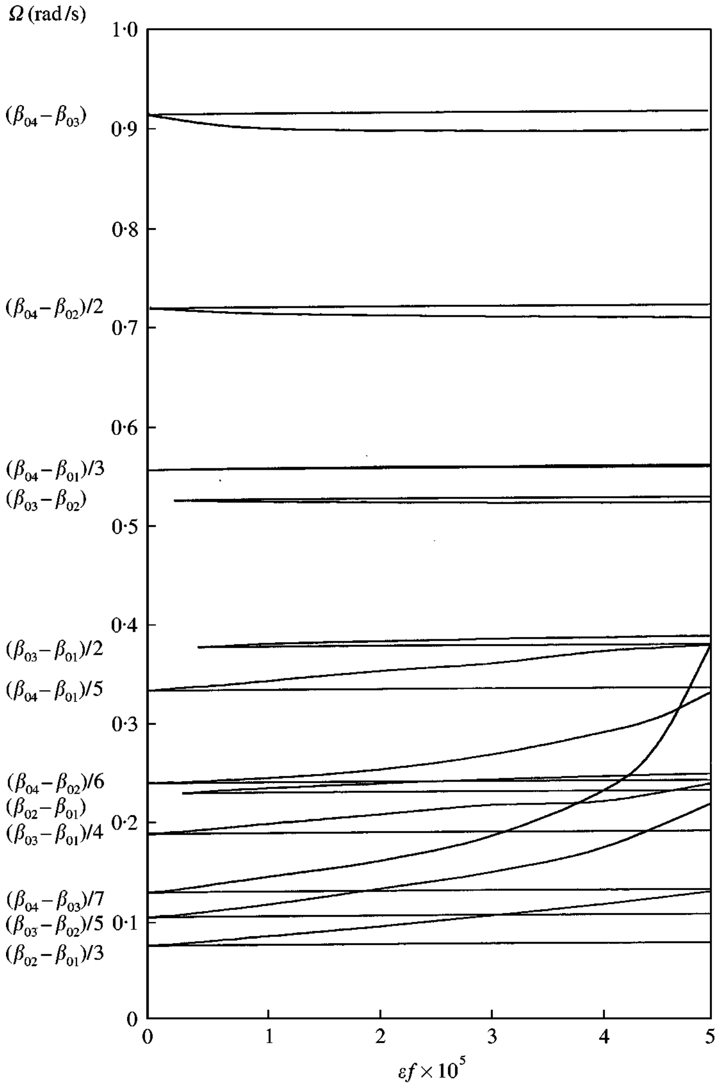

Figure 3. Regions of instability of friction-induced combination resonances.

Note that the backward travelling waves marked by * in Table 1 are stable only in the range of friction of $0 \leqslant \varepsilon f \leqslant 5 \times 10^{-5}$ shown in Figure 3 and these backward travelling waves can still be made unstable by greater friction of $\varepsilon f > \varepsilon\varsigma(\beta_{rs} + s\Omega)/s$ $(s > 0)$ since this greater friction values make positive the real part of the exponent of A_{rs} in equations (21) and (23). Similarly, greater values of friction of $\varepsilon f > \varepsilon\varsigma(\beta_{kl} + l\Omega)/l$ $(l > 0)$ make A_{kl} in equation (23) divergent. The unstable (k, l) backward wave marked by $\Leftarrow$ has a larger positive real part in its exponent than that of (r, s) forward wave.

The above results are interesting in particular to the disc brake squeal problem where a number of squeal mechanisms [23], including the follower-force mechanism, have been put forward. Finally it should be noted that those models of disc vibration which do not take into account the relative rotation between the disc and the auxiliary system, like the one proposed in reference [24], are incapable of predicting combination resonances.

6. CONCLUSIONS

This paper has investigated the instability of the forward and backward travelling waves in the transverse vibration of a stationary disc induced by the friction in a rotating mass–spring–damper system. Combination resonances have been considered across the whole speed range and the friction force has been modelled as a follower force. A simulated example has been presented to show the regions of instability for combination resonances and the instability of the associated travelling waves at the subcritical speeds. In addition to the already-known facts that the friction alone destabilizes the backward travelling waves in the speed-independent resonances and that damping of the rotating system is destabilizing in the supercritical speed range, it was found in this investigation that the friction modelled as a follower force is the most destabilizing factor among the system parameters. The findings made in this paper may be useful in validating the mechanisms for unstable vibrations in discs. Friction can destabilize the backward travelling waves of both modes in all the summation-type resonances in the subcritical range. One of the modes (k, l) is unstable in the range of friction shown in Figure 3 whereas the other (r, s) only becomes unstable at higher levels of friction. In the difference-type of combinations, which generally occur at higher subcritical speeds, the regions of instability are narrower (as shown in Figure 3) so they are less important than the summation-type combinations. Friction is destabilizing to both the forward waves and the backward ones in both modes of the combination.

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APPENDIX A: NOMENCLATURE

a, b	inner and outer radii of the disc
c, k, m	damping, stiffness, mass of the rotating system respectively
h	thickness of the disc
i	$\sqrt{-1}$
$q_{kl}(q_{rs})$	modal co-ordinate for $k(r)$ nodal circles and $l(s)$ nodal diameters for the disc
r	radial co-ordinate in cylindrical co-ordinate system
r_0	initial radial position of the rotating system
t	time
w	deflection of the disc in cylindrical co-ordinate system
D	flexural rigidity

D^*	Kelvin-type damping coefficient
E	Young's modulus
F	the friction force between the disc and the rotating system in the circumferential direction, $F = \mu N$
G_{kl}, H_{kl}	constants of integration in the solution of q_{kl}
N	twice the normal load due to brake pressure
R_{kl}	combination of Bessel functions to represent the mode shape of the disc in the radial direction
$\delta(\cdot)$	Dirac delta function
δ_{kl}	Kronecker delta
θ	circumferential co-ordinate in cylindrical co-ordinate system
λ	characteristic exponent in $\exp(i\lambda\tau)$ which describes the dynamic response of the disc in the time domain
v	the Poisson ratio of the disc material
ξ_K	damping coefficient ($= D^*/2D$) of the disc
μ	friction coefficient
ρ	mass-density of the disc
ψ_{kl}	mode shape function for the transverse vibration of the disc corresponding to q_{kl}
σ	detuning parameter
ω_{kl}	natural (circular) frequency corresponding to q_{kl}
$\tilde{\Omega}$	constant rotating speed of the disc in radians per second